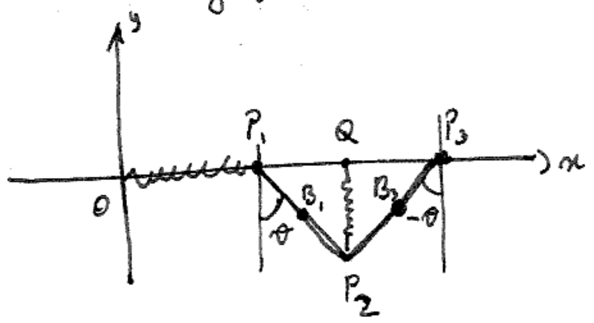
Esame Giugno 2003



- $P_1\hat{P_2}P_3$ triangolo isoscele

  $\overline{P_1P_2} = \overline{P_2P_3} = \ell$
- massa $\overline{P_1P_2} = M$ = massa $\overline{P_2P_3}$
- massa $Q = 0$
- $k_1 = \widetilde{OP_1}$, $k_2 = \widetilde{QP_2}$

1) scelgo le coord. libere $x = x_Q = x(t)$, $\vartheta = \vartheta(t)$

$Q = (x, 0)$ $\quad P_1 = (x - \ell\sin\vartheta, 0)$ $\quad P_3 = (x + \ell\sin\vartheta, 0)$

$P_2 = (x, -\ell\cos\vartheta)$ $\quad B_1 = (x - \frac{\ell}{2}\sin\vartheta, -\frac{\ell}{2}\cos\vartheta)$ $\quad B_3 = (x + \frac{\ell}{2}\sin\vartheta, -\frac{\ell}{2}\cos\vartheta)$

$\underline{v}_{B_1} = \left(\dot x - \frac{\ell}{2}\dot\vartheta\cos\vartheta, \frac{\ell}{2}\dot\vartheta\sin\vartheta\right)$ $\quad v_{B_3} = \left(\dot x + \frac{\ell}{2}\dot\vartheta\cos\vartheta, \frac{\ell}{2}\dot\vartheta\sin\vartheta\right)$

2) Lagrangiana e eq. di Lagrange

$$T = \frac{1}{2}M\left(\underline{v}_{B_1}\cdot\underline{v}_{B_1} + \underline{v}_{B_3}\cdot\underline{v}_{B_3}\right) + \frac{1}{2}I_{\overline{P_1P_2}}\dot\vartheta^2 + \frac{1}{2}I_{\overline{P_2P_3}}\dot\vartheta^2$$

$$I_{P_1P_2} = I_{P_2P_3} = \delta\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} d\rho\,\rho^2 = \frac{M}{\ell}\left.\frac{\rho^3}{3}\right|_{-\ell/2}^{\ell/2} = \frac{M}{\ell}\frac{2\ell^3}{24} = \frac{M\ell^2}{12}$$

$$v_{B_1}\cdot v_{B_1} = \dot x^2 + \frac{\ell^2}{4}\dot\vartheta^2\cancel{\cos^2\vartheta} - \ell\dot x\dot\vartheta\cos\vartheta + \cancel{\frac{\ell^2}{4}\dot\vartheta^2\sin^2\vartheta}$$

$$v_{B_3}\cdot v_{B_3} = \dot x^2 + \frac{\ell^2}{4}\dot\vartheta^2\cancel{\cos^2\vartheta} + \ell\dot x\dot\vartheta\cos\vartheta + \cancel{\frac{\ell^2}{4}\dot\vartheta^2\sin^2\vartheta}$$

$$T = \frac{1}{2}M\,2\dot x^2 + \frac{M\ell^2}{4}\dot\vartheta^2 + \frac{M\ell^2}{12}\dot\vartheta^2 = M\dot x^2 + \frac{M\ell^2}{3}\dot\vartheta^2$$

$$U = Mg\,y_{B_1} + Mg\,y_{B_3} + \frac{1}{2}k_1\overline{OP_1}^2 + \frac{1}{2}k_2\overline{QP_2}^2 =$$

$$= -Mg\ell\cos\vartheta + \frac{1}{2}k_1x^2 + \frac{1}{2}k_1\ell^2\sin^2\vartheta - k_1x\ell\sin\vartheta + \frac{1}{2}k_2\ell^2\cos^2\vartheta$$

$$L = T - U = M\dot x^2 + \frac{M\ell^2}{3}\dot\vartheta^2 + Mg\ell\cos\vartheta - \frac{1}{2}k_1x^2 - \frac{1}{2}k_1\ell^2\sin^2\vartheta + k_1x\ell\sin\vartheta - \frac{1}{2}k_2\ell^2\cos^2\vartheta$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot x} - \dfrac{\partial L}{\partial x} = 2M\ddot x + k_1x - k_1\ell\sin\vartheta = 0 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot\vartheta} - \dfrac{\partial L}{\partial\vartheta} = \dfrac{2}{3}M\ell^2\ddot\vartheta + Mg\ell\sin\vartheta + k_1\ell^2\sin\vartheta\cos\vartheta - k_1\ell x\cos\vartheta - k_2\ell^2\sin\vartheta\cos\vartheta = 0 \end{cases}$$

3) punti di equilibrio

$$\begin{cases} 1) \ \dfrac{\partial U}{\partial x} = k_1 (x - l \sin\vartheta) = 0 \quad = k_1 x - k_1 l \sin\vartheta \\ 2) \ \dfrac{\partial U}{\partial \vartheta} = Mgl \sin\vartheta - k_1 l x \cos\vartheta + \dfrac{k_1 l^2}{2} \sin 2\vartheta - \dfrac{k_2 l^2}{2} \sin 2\vartheta = 0 \end{cases}$$

1) $\Rightarrow$ $x = l \sin\vartheta$

$\Downarrow$

2) $Mgl \sin\vartheta - k_1 \cancel{l^2 \sin\vartheta \cos\vartheta} + \cancel{\dfrac{k_1 l^2}{2} \sin 2\vartheta \cos\vartheta} - k_2 l^2 \sin\vartheta \cos\vartheta =$

$= l \sin\vartheta \left(Mg - k_2 l \cos\vartheta\right) = 0$

$\Leftrightarrow \sin\vartheta = 0 \quad \vee \quad \cos\vartheta = \dfrac{Mg}{k_2 l}$

$\sin\vartheta = 0 \to \vartheta = 0, \pi \to x = 0$

$\cos\vartheta = \dfrac{Mg}{k_2 l} \to \vartheta = \pm\beta = \arccos \dfrac{Mg}{k_2 l}$

purché $\dfrac{Mg}{k_2 l} \leq 1$

$$A(x,\vartheta) = \mathrm{Hess}\, U = \begin{pmatrix} k_1 & -k_1 l \cos\vartheta \\ -k_1 l \cos\vartheta & Mgl\cos\vartheta + k_1 l x \sin\vartheta + k_1 l^2 \cos 2\vartheta - k_2 l^2 \cos 2\vartheta \end{pmatrix}$$

1) $A(0,0) = \begin{pmatrix} k_1 & -k_1 l \\ -k_1 l & Mgl + k_1 l^2 - k_2 l^2 \end{pmatrix}$

$\det A(0,0) = k_1 (Mgl + \cancel{k_1 l^2} - k_2 l^2) - \cancel{k_1^2 l^2} = k_1 l (Mg - k_2 l) \gtrless 0$

per $\dfrac{Mg}{k_2 l} \gtrless 1$

- > 1 considerando che $k_1 > 0$ è un termine diag. $\Rightarrow$ A def. positiva $\Rightarrow$ 2 aut. pos. $\Rightarrow$ stab.
- < 1 $\to$ 1 autoval < 0 $\Rightarrow$ inst.
- $= 1$ $\to$ ho $\lambda_1 = 0$ e $k_1 > 0$ $\Rightarrow$ BOH!

$$-A(0,\pi) = \begin{pmatrix} k_1 & k_1 l \\ k_1 l & -M_g l + k_1 l^2 - k_2 l^2 \end{pmatrix}$$

$$\det(A(0,\pi)) = k_1 l\left(-M_g + \cancel{k_1 l} - k_2 l\right) - \cancel{k_1^2 l^2} =$$

$$= -k_1 l\left(M_g + k_2 l\right) \gtrless 0$$

$$\cancel{M_g + k_2 l \lessgtr 0}$$

$-M_g - k_2 l < 0$ sempre $\Rightarrow$ inst.

$$-A(l\sin\beta,\beta) = \begin{pmatrix} k_1 & -k_1 l\cos\beta \\ -k_1 l\cos\beta & M_g l\cos\beta + k_1 l^2\sin^2\beta + k_1 l^2\cos 2\beta - k_2 l^2\cos 2\beta \end{pmatrix}$$

$$\downarrow$$

$$\frac{M_g^2}{k_2} + k_1 l^2\sin^2\beta + 2k_1 l^2\cos^2\beta - k_1 l^2 - 2k_2 l^2\cos^2\beta + k_2 l^2$$

$$\cos 2x = \frac{e^{i2x} + e^{-i2x}}{2} = \frac{(e^{ix})^2 + (e^{-ix})^2 + 2 - 2}{2} =$$

$$= 2\left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 - 1 = 2\cos^2 x - 1 \left(= 1 - \sin^2 x\right)$$

$$= \frac{M_g^2}{k_2} + \cancel{k_1 l^2} + k_1 l^2\cos^2\beta + \cancel{2k_1 l^2\cos^2\beta} - \cancel{k_1 l^2} - 2k_2 l^2\cos^2\beta + k_2 l^2$$

$$\det A(l\sin\beta,\beta) = k_1\frac{M_g^2}{k_2} + \cancel{k_1^2 l^2\cos^2\beta} - 2k_1 k_2 l^2\cos^2\beta + k_1 k_2 l^2 - \cancel{k_1^2 l^2\cos^2\beta} =$$

$$= k_1 l\left(\frac{M_g^2}{k_2 l} - 2\cancel{k_2} l\frac{M_g^2}{k_2^2 l^2} + k_2 l\right) = \frac{k_1 l}{k_2 l}\left(-M_g^2 + k_2^2 l^2\right) \gtrless 0$$

$$\Leftrightarrow\ -\frac{M_g^2}{k_2^2 l^2} \gtrless -1 \ \Leftrightarrow\ \frac{M_g^2}{k_2 l^2} \lessgtr 1$$

>1 IMP.

$=1$ + k_1 (elem. diag >0) $\Rightarrow$ BOH!

<1 + k_1 (elem. diag >0) $\Rightarrow$ 2 aut. >0 $\Rightarrow$ STAB.

$$-A(-l\sin\beta,-\beta)=\begin{pmatrix} k_1 & -k_1 l\cos\beta \\ -k_1 l\cos\beta & Mgl\cos\beta+k_1 l^2\sin^2\beta+k_1 l^2\cos 2\beta-k_2 l^2\cos 2\beta\end{pmatrix}$$

$\cos\beta=\cos(-\beta)$

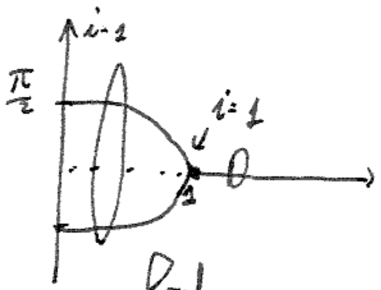
$\sin(-\beta)=-\sin\beta$

$$=A(l\sin\beta,\beta)$$

Caso $\frac{Mg}{k_2 l}=1 \Rightarrow \beta=\arccos 1=0$

$(\pm l\sin\beta,\pm\beta)=(0,0)$ dunque ho 1 solo caso

inoltre $\lim_{\frac{Mg}{k_2 l}\to 0}\beta=\frac{\pi}{2}$



i = indice di stabilità

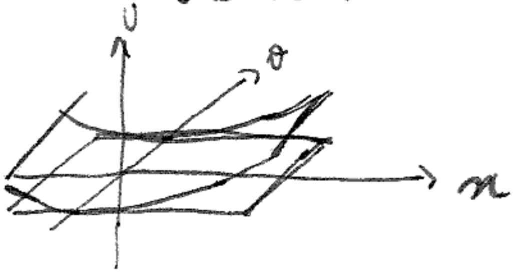
~~...~~ congettura: il mio punto è stabile (conservo l'indice di stabilità

biforcazione a forchetta

se sto in $(0,0)$ sono ~~in pos~~ sulla verticale

$$U=-Mgl\cos\vartheta+\frac{1}{2}k_1x^2+\frac{1}{2}kl^2\sin^2\vartheta-k_1xl\sin\vartheta+\frac{1}{2}k_2l^2\cos^2\vartheta$$

in x è parabolico, in uno è come un po' ballerino ma limitato



$\lim_{|x|\to\infty}\inf_{\vartheta\in[0,2\pi)}=+\infty$

il min deve esistere al finito

ho 2 p.ti critici: $(0,0)$ e $(0,\pi)$

Morse: $(0,\pi)$ è sella $\Rightarrow$ lasso

~~A~~ $(0,0)$ ha aut. $>0 \Rightarrow$ L.-D. $\Rightarrow$ stabile

3) $k_1 = 0$ det. cost. del moto

riscrivo la Lagrangiana

$$\mathcal{L} = M\dot{x}^2 + \frac{Ml^2}{3}\dot{\vartheta}^2 + Mgl\cos\vartheta - \frac{1}{2}k_2 l^2 \cos^2\vartheta$$

oss $\frac{\partial \mathcal{L}}{\partial x} = 0$ ~~⇒~~ x è coord. ciclica

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = c$$ è costante del moto

$$2M\dot{x} = c$$

I eq. cardinale, baricentro $B_1 = (x, -\frac{l}{2}\cos\vartheta)$

$$2M\ddot{x}_{B_1} = \underline{R}^{(ext)} = -k_2\overrightarrow{QP_2} - \underset{\substack{\uparrow \\ 2\text{ aste}}}{2Mg}\,\underline{e}_y$$

oss $\underline{R}^{(ext)} \parallel \underline{e}_y$

$$\underline{R}^{(ext)} \cdot \underline{e}_x = 0$$

$\Rightarrow 2M\dot{x}$, quantità di moto totale lungo $\underline{e}_x$, si conserva

Si conserva anche $E = T + U$

4) c.i. $Mg = k_2 l$ $\quad k_1 = 0$ $\quad Q(0) = 0 \Rightarrow (x(0) = 0)$

$$v_{P_1}(0) + v_{P_2}(0) = 0 \Rightarrow 2\dot{x}(0) = 0$$

$$\dot{x} - \dot{\vartheta} l \cos\vartheta + \dot{x} = 0 \Rightarrow 2\dot{x}(0) - \dot{\vartheta}(0)\, l\cos\vartheta(0) = 0$$

$$2M\dot{x}(t) = c = 0 \Rightarrow \dot{x}(t) = 0 \Rightarrow x(t) = c = x(0) = 0$$

$x(t) = 0 \ \forall t$ è soluzione del moto

in $\vartheta(t)$: $\frac{2}{3}Ml^2\ddot{\vartheta} + Mgl\sin\vartheta - k_2 l^2 \sin\vartheta\cos\vartheta$

$$\frac{2}{3}Ml^2\ddot{\vartheta} = -Mgl\sin\vartheta + k_2 l^2 \sin\vartheta\cos\vartheta$$

$\frac{Mg}{k_2 l}$ / (x,ϑ)	<1	$=1$	>1
$(0,0)$	INST.	~~BOH~~. STAB. per y^4	STAB
$(0,\pi)$	INST.	INST.	INST.
$(l\sin\beta, \beta)$	STAB.	~~BOH~~. STAB. per y^4	$\nexists$
$(-l\sin\beta, \beta)$	STAB.	~~BOH~~. STAB. per y^4	$\nexists$

5) $k_1 > 0 \quad \gamma = 0 \qquad$ vel. ang. $\Omega\, e_y \neq 0$

$P_2(0) = Q \quad (\Rightarrow \vartheta(0) = \frac{\pi}{2})$

$v_{P_2}(0) \cdot e_y = 0 \quad (\Rightarrow l\, \dot\vartheta \sin\vartheta(0) = 0)$

ma $\vartheta(0) = 0 \Rightarrow \dot\vartheta(0) = 0$

dire Ω t.c.

$$\lim_{t\to+\infty} P_1(t) = \lim_{t\to+\infty} P_2(t) = \lim_{t\to+\infty} P_3(t) = +\infty$$

Th Coriolis: $\quad \underline{a}_a = \underline{a}_t + \underline{a}_R + \underline{a}_c$

$$a_T = \cancel{a_{O'}} + \dot\omega \cancel{\wedge \overrightarrow{O'P}} + \omega \wedge (\omega \wedge \overrightarrow{O'P})$$

O' fisso.

$a_c = 2\underline{\omega} \wedge \underline{v}_R \quad \perp$ al piano, bilanciata dal vincolo

le forze apparenti sono date da $m\omega \wedge (\omega \wedge \overrightarrow{O'P}) \quad \forall P$ del sistema con massa m

$= m\Omega^2 \overrightarrow{P_* P}$

$$U_{centr.} = -\frac{m}{2}\Omega^2 \overline{P_* P}^2 \simeq \sum -m_j \frac{\Omega^2}{2} x_j^2 = -\int_0^l \frac{\delta\Omega^2}{2} d\rho\, (x - \rho\sin\vartheta)^2 + -\int_0^l \frac{\delta\Omega^2}{2} d\rho\, (x + \rho\sin\vartheta)^2$$

$$= -\delta\Omega^2 x^2 l - \frac{\delta\Omega^2}{3} l^3 \sin^2\vartheta$$

riscrivo la lagrangiana

$$\tilde{\mathcal{L}} = M\dot{x}^2 + \frac{Ml^2}{3}\dot{\vartheta}^2 - \frac{k_1}{2}x^2 + k_1 l x \sin\vartheta - \frac{k_1 l^2}{2}\sin^2\vartheta - \frac{k_1 l^2}{2}\cos^2\vartheta + M\Omega^2 x^2 + \frac{Ml^2\Omega^2}{3}\sin^2\vartheta$$

$$\frac{d}{dt}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\vartheta}}$$