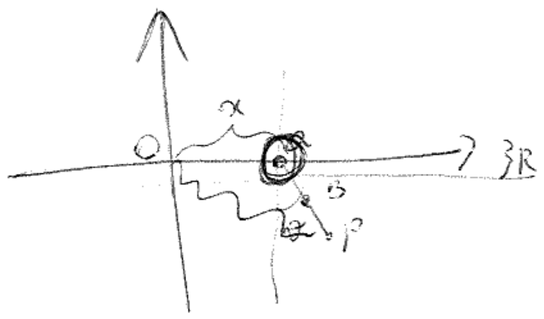
Esame 4/2/2003



- Disco omogeneo con baricentro Q e raggio R su guida a $-R=y$
- Asta omogenea $\overrightarrow{QP}$ di lunghezza l e massa m
- k tra O, P

Coord lagrangiane x, ϑ

$x \uparrow x_Q$ $\quad \vartheta \uparrow P\hat{Q}y$

$Q = (x, 0) \qquad P = (x + l \sin\vartheta, -l\cos\vartheta)$

$B = (\text{baricentro asta}) = \left(x + \frac{l}{2}\sin\vartheta, -\frac{l}{2}\cos\vartheta\right)$

$v_Q = (\dot{x}, 0), \quad v_P = (\dot{x} + l\dot{\vartheta}\cos\vartheta, +l\dot{\vartheta}\sin\vartheta)$

$v_B = \left(\dot{x} + \frac{l}{2}\dot{\vartheta}\cos\vartheta, \frac{l}{2}\dot{\vartheta}\sin\vartheta\right)$

$T_{DISCO} = \frac{1}{2} M \underline{v_Q}\,\underline{v_Q} + \frac{1}{2} I_{DISCO}\,\omega^2$

MOMENTO ANG. DISCO $\quad \sum m_j QD_j \wedge v_j = \sum m_j QD_j \wedge \omega \wedge QD_j =$

$= \sum m_j QD_j \cdot \omega e_z = \sum m_j (x_j^2 + y_j^2)\,\omega^2$

E_{CIN} DI SPIN $= \frac{1}{2} \sum m_j (x_j^2 + y_j^2)\,\omega^2$

$I_{DISCO} := \delta \int_0^R d\rho \; 2\pi \rho \, \rho^2 = \delta\pi \frac{R^2 R^2}{2} = M\frac{R^2}{2}$

IL DISCO ROTOLA SENZA STRISCIARE

$T_{DISCO} = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} \frac{MR^2}{2} \frac{\dot{x}^2}{R^2}$

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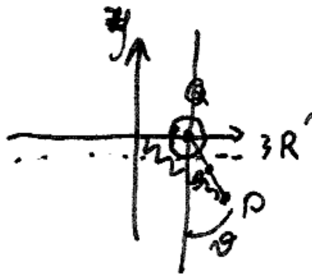

- disco omogeneo, raggio R, centro massa M Q guida a $-R$
- asta omogenea $\overrightarrow{QP}$ lungh $= \ell$ massa m
- molla di cost k tra O e P

x, ϑ coord. lagrangiane

$x = x_Q$, $\vartheta = P\hat{Q}y$

$Q = (x, 0)$ $\qquad \underline{v}_Q = (\dot x, 0)$

$P = (x + \ell \sin\vartheta, -\ell\cos\vartheta)$ $\qquad \underline{v}_P = (\dot x + \ell\dot\vartheta\cos\vartheta, \ell\dot\vartheta\sin\vartheta)$

$B = (x + \frac{\ell}{2}\sin\vartheta, -\frac{\ell}{2}\cos\vartheta)$ $\qquad \underline{v}_B = (\dot x + \frac{\ell}{2}\dot\vartheta\cos\vartheta, \frac{\ell}{2}\dot\vartheta\sin\vartheta)$

$$T_{DISCO} = \underbrace{\frac12 M\,\underline{v}_Q\cdot\underline{v}_Q}_{= \frac12 M\dot x^2} + \frac12 I_{DISCO}\,\omega^2 = \frac12 M\dot x^2 + \frac12\frac{M}{2}\dot x^2 = \frac34 M\dot x^2$$

$I_{DISCO} = \frac12 M R^2 \qquad \omega = \frac{v}{R} = \frac{\dot x}{R}$

$$T_{asta} = \frac12 m\,\underline{v}_B\cdot\underline{v}_B + \frac12 I_{ASTA}\,\dot\vartheta^2 = \frac12 m\dot x^2 + \frac16 m\ell^2\dot\vartheta^2 + \frac12 m\ell\dot x\dot\vartheta\cos\vartheta$$

$$I_{ASTA} = \delta\int_{-\ell/2}^{\ell/2} d\rho\,\rho^2 = \delta\,\frac{\rho^3}{3}\Big|_{-\ell/2}^{\ell/2} = \frac13\delta\frac{2\ell^3}{8} = \frac{1}{12}m\ell^2$$

$\delta = \frac{m}{\ell}$

$$\left(\dot x + \frac{\ell}{2}\dot\vartheta\cos\vartheta, \frac{\ell}{2}\dot\vartheta\sin\vartheta\right)\begin{pmatrix}\dot x + \frac{\ell}{2}\dot\vartheta\cos\vartheta \\ \frac{\ell}{2}\dot\vartheta\sin\vartheta\end{pmatrix} = \dot x^2 + \ell\dot x\dot\vartheta\cos\vartheta + \frac{\ell^2}{4}\dot\vartheta^2\cos^2\vartheta + \frac{\ell^2}{4}\dot\vartheta^2\sin^2\vartheta = \dot x^2 + \ell\dot x\dot\vartheta\cos\vartheta + \frac{\ell^2}{4}\dot\vartheta^2$$

$$\frac12 m\frac{\ell^2}{4}\dot\vartheta^2 + \frac12\frac{m}{12}\ell^2\dot\vartheta^2 = \frac{1}{24}(3m\ell^2\dot\vartheta^2 + m\ell^2\dot\vartheta^2) = \frac{m}{6}\ell^2\dot\vartheta^2$$

$\cancel{T = \frac34\frac12\left(\frac32 M + m\right)}$

$$T = \frac34 M\dot x^2 + \frac12 m\dot x^2 + \frac16 m\ell^2\dot\vartheta^2 + \frac12 m\ell\dot x\dot\vartheta\cos\vartheta = \frac12\left(\frac32 M + m\right)\dot x^2 + \frac16 m\ell^2\dot\vartheta^2 + \frac12 m\ell\dot x\dot\vartheta\cos\vartheta$$

$$U = m g\, y_B + \frac12 k\|\overrightarrow{OP}\|^2 = -mg\frac{\ell}{2}\cos\vartheta + \frac12 k(x^2 + 2\ell x\sin\vartheta + \ell^2\sin^2\vartheta + \ell^2\cos^2\vartheta) =$$
$$= -mg\frac{\ell}{2}\cos\vartheta + \frac12 k(x^2 + 2\ell x\sin\vartheta + \underbrace{\ell^2}_{\text{costante}})$$

$$L = T - U = \frac12\left(\frac32 M + m\right)\dot x^2 + \frac16 m\ell^2\dot\vartheta^2 + \frac12 m\ell\dot x\dot\vartheta\cos\vartheta + mg\frac{\ell}{2}\cos\vartheta - \frac12 k x^2 - k\ell x\sin\vartheta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot x} = \left(\frac32 M + m\right)\ddot x + \frac12 m\ell\ddot\vartheta\cos\vartheta - \frac12 m\ell\dot\vartheta^2\sin\vartheta \ ; \quad \frac{\partial L}{\partial x} = -kx - k\ell\sin\vartheta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\vartheta} = \frac13 m\ell^2\ddot\vartheta + \frac12 m\ell\ddot x\cos\vartheta - \frac12 m\ell\dot\vartheta\dot x\sin\vartheta \ ; \quad \frac{\partial L}{\partial\vartheta} = -\frac12 m\ell\dot x\dot\vartheta\sin\vartheta - mg\frac{\ell}{2}\sin\vartheta - k\ell x\cos\vartheta$$

$$\frac{\partial L}{\partial \dot{x}} = \left(\frac{3}{2}M+m\right)\dot{x} + \frac{1}{2} m l \dot{\vartheta} \cos\vartheta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \left(\frac{3}{2}M+m\right)\ddot{x} + \frac{1}{2} m l \ddot{\vartheta} \cos\vartheta - \frac{1}{2} m l \dot{\vartheta}^2 \sin\vartheta$$

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$$\frac{\partial L}{\partial x} = -kx - kl\sin\vartheta$$

$$\frac{\partial L}{\partial \dot{\vartheta}} = \frac{1}{3} m l^2 \dot{\vartheta} + \frac{1}{2} m l \dot{x}\cos\vartheta \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} = \frac{1}{3} m l^2 \ddot{\vartheta} + \frac{1}{2} m l \ddot{x}\cos\vartheta - \frac{1}{2} m l \dot{x}\dot{\vartheta}\sin\vartheta$$

$$\frac{\partial L}{\partial \vartheta} = -\frac{1}{2} m l \dot{x}\dot{\vartheta}\sin\vartheta - mg\frac{l}{2}\sin\vartheta - klx\cos\vartheta$$

$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \left(\frac{3}{2}M+m\right)\ddot{x} + \frac{1}{2} m l \ddot{\vartheta}\cos\vartheta - \frac{1}{2} m l \dot{\vartheta}^2\sin\vartheta + kx + kl\sin\vartheta = 0 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} - \frac{\partial L}{\partial \vartheta} = \frac{1}{3} m l^2 \ddot{\vartheta} + \frac{1}{2} m l \ddot{x}\cos\vartheta \cancel{- \frac{1}{2} m l \dot{x}\dot{\vartheta}\sin\vartheta} \cancel{+ \frac{1}{2} m l \dot{x}\dot{\vartheta}\sin\vartheta} + mg\frac{l}{2}\sin\vartheta + klx\cos\vartheta = 0 \end{cases}$$

2) det. punti di equilibrio

$$\nabla U = 0 \iff \begin{cases} \frac{\partial U}{\partial x} = kx + kl\sin\vartheta = 0 \\ \frac{\partial U}{\partial \vartheta} = mg\frac{l}{2}\sin\vartheta + klx\cos\vartheta = 0 \end{cases}$$

$$\text{Hess}\, U = \begin{pmatrix} k & kl\cos\vartheta \\ kl\cos\vartheta & mgl\cos\vartheta - klx\sin\vartheta \end{pmatrix}$$

ho $\nabla U = 0$ per $kx = -kl\sin\vartheta$; $x = -l\sin\vartheta$

$$\Rightarrow \frac{\partial U}{\partial \vartheta} = mg\frac{l}{2}\sin\vartheta - kl^2\sin\vartheta\cos\vartheta = \sin\vartheta\left(mg\frac{l}{2} - kl^2\cos\vartheta\right) = 0$$

dunque ho $\sin\vartheta = 0 \iff \vartheta = 0, \pi$

$\vee$

$mg = 2kl\cos\vartheta \iff \cos\vartheta = \frac{mg}{2kl} \iff \vartheta = \pm\beta$, $\beta := \arccos\frac{mg}{2kl}$

perché $\frac{mg}{2kl} \in [0,1]$

$\nabla U = (0,0)$ per

$\vartheta = 0 \to \text{Hess}\, U|_{(0,0)}$

$\vartheta = \pi \to \text{Hess}\, U|_{(0,\pi)}$

$\vartheta = \beta \to \text{Hess}\, U|_{(-l\sin\beta,\beta)}$

$\vartheta = -\beta \to \text{Hess}\, U|_{(-l\sin(-\beta),-\beta)}$, $= l\sin\beta$

$$\nabla U = \left(kx + kl \sin\vartheta,\ \frac{mg}{2} l \sin\vartheta + klx\cos\vartheta \right)$$

$$\mathrm{Hess}\, U = \begin{pmatrix} k & kl\cos\vartheta \\ kl\cos\vartheta & \frac{mg}{2} l\cos\vartheta - klx\sin\vartheta \end{pmatrix}$$

$$\mathrm{Hess}\, U \Big|_{(-l\sin\beta,\,\beta)} = \begin{pmatrix} k & kl\cos\beta \\ kl\cos\beta & \frac{mg}{2} l\cos\beta + kl^2\sin\beta \cancel{\sin\beta} \end{pmatrix}$$

$$\det \mathrm{Hess}\, U\Big|_{(-l\sin\beta,\beta)} = \frac{mg}{2} kl\cos\beta + k^2l^2\sin^2\beta \cancel{\cos\beta} - k^2l^2\cos^2\beta =$$

~~traccia = k + m²g²l / 2 kl~~

$$= \frac{mg}{2} kl \frac{mg}{2kl} + k^2l^2(1-\cos^2\beta) - k^2l^2\cos^2\beta =$$

$$= \left(\frac{m^2g^2}{\cancel{4}}\right)^2 + k^2l^2 - k^2l^2\cos^2\beta - k^2l^2\cos^2\beta =$$

$$= \left(\frac{mg}{2}\right)^2 + k^2l^2 - 2k^2l^2\left(\frac{mg}{2kl}\right)^2 =$$

$$= \frac{m^2g^2}{4} + k^2l^2 - 8\cancel{k^2l^2}\frac{m^2g^2}{24\cancel{k^2l^2}} = k^2l^2 - \frac{m^2g^2}{4} =$$

$$= \underset{>0}{kl}\left(kl - \left(\frac{mg}{2}\right)^2\frac{1}{kl}\right) \gtrless 0$$

$$\Leftrightarrow \quad 1 - \frac{m^2g^2}{4k^2l^2} \gtrless 0 \quad \Leftrightarrow \quad \left|\overset{>0}{\frac{mg}{2kl}}\right| \lesseqgtr 1 \Leftrightarrow \frac{mg}{2kl} \lesseqgtr 1$$

$-l\sin\beta, \beta$ è ~~inst~~ (STAB. >0) se $\frac{mg}{2kl} < 1$

BOH se $\frac{mg}{2kl} = 1$

$\nexists$ se $\frac{mg}{2kl} > 1$ (non i β)

tr: $k + \frac{m^2g^2 \cancel{l}}{2\,kl} + kl^2 - \cancel{k}\cancel{l}\frac{m^2g^2}{4k^2l^4}$ ~~> instab~~ ~~$\lambda_1 > 0$~~ ~~$\lambda_2 > 0$~~

NON SO

$$\mathrm{Hess}\, U\Big|_{(l\sin\beta,\,-\beta)} = \begin{pmatrix} k & kl\cos\beta \\ kl\cos\beta & \frac{mgl}{2}\cos\beta - kl^2\sin\beta\sin\beta \end{pmatrix} *$$

$$\det \mathrm{Hess}\, U\Big|_{(\ )} = \frac{mgl}{2}k\cos\beta - k^2l^2\cancel{\sin^2\beta} + \cancel{k^2l^2\cos^2\beta} - k^2l^2\cancel{\cos^2\beta} =$$

$$= \frac{m^2g^2}{4} - k^2l^2 \gtrless 0$$

$$\frac{mg}{2kl} \gtrless 1$$

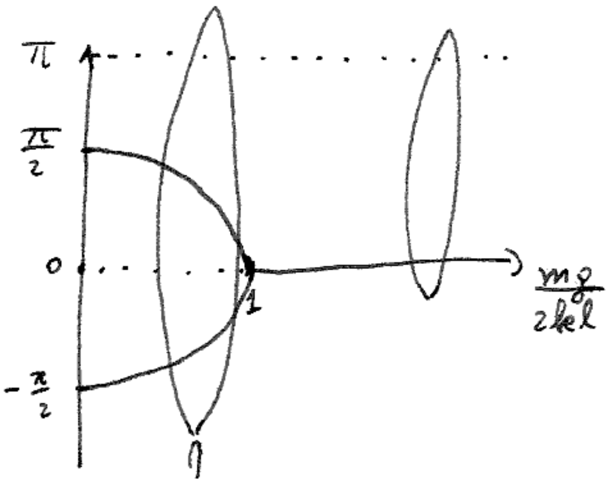
>1 ~~STAB~~ $\nexists$

$=1$ BOH

<1 INST

Ora dobbiamo affrontare il caso ~~$\beta=1$~~ $\frac{mg}{2kl}=1$ nel dettaglio

in questo caso ho $\cos\vartheta = \frac{1}{1}$, cioè $\vartheta = \pm\frac{\pi}{2}$



l'indice è 0, deve conservarsi

sottraendo -2, aggiungo 2 per conservare $\Rightarrow$ $\frac{mg}{2kl} < 1$ e β fino a $\frac{\pi}{2}$ è stabile

per $\frac{mg}{2kl} = 1$, $\vartheta=0$ ho

$U = -mg\frac{l}{2} + \frac{1}{2}kx^2 + \cancel{kl}$ ← è costante!

ma O è stabile